

An Exact Solution of a Generalization of the Rabi Model

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There has been renewed theoretical interest recently in the Rabi model due to Braak's analytical solution and introduction of a new criterion for integrability. We focus not on the integrability of the system but rather why it is solvable in the first place. We show that the Rabi model is the limiting case of a more general finite dimensional system by use of a contraction and suggest that this is the reason for its solvability, which still applies in the case of non-integrable but solvable variations.

I. INTRODUCTION

In classical mechanics the terms integrable and solvable are often used interchangeably, implying that a system that is integrable is also said to be solvable while a non-integrable system exhibits chaos. In this sense it became clear in the early twentieth century that most classical dynamical systems are not analytically solvable, for example the perturbation series for the three body problem is only convergent in certain regions of the phase space. Later work by Kolmogorov, Arnold and Moser [1] established that this region covers a large volume for small perturbations but has a complicated fractal structure. Thus, chaos and instability are still possible for small bodies in nearly Keplerian orbits and the solar system appears stable because such bodies were either kicked out or fell into the Sun or Jupiter.

An analogous understanding still does not exist in the case of quantum mechanics. The superficial observation that the problem to be solved is linear (the Schrodinger equation) misses the point that the Hilbert spaces of most systems of interest are infinite dimensional: a linear problem in infinite dimensions has many of the analytical subtleties of finite dimensional non-linear ordinary differential equations [2]. One way to approach the question of solvability of a quantum system is to ask if it has sensible finite dimensional truncations which are exactly solvable.

Recently Braak [3] analytically solved the Rabi model [4], with Hamiltonian

$$H_R = \omega a^\dagger a + \frac{\Delta}{2} \sigma_3 + g \sigma_1 (a + a^\dagger), \quad (1)$$

which can be thought of as a two level atom with energy levels $\pm\Delta/2$ coupled to a quantized field with frequency ω . As Braak mentions, the general opinion was that this system is not solvable because the Hamiltonian is its only conserved quantity. A second conserved quantity would allow the Hamiltonian to be brought to diagonal form. For example, [5] in the Jayne-Cummings model $Q = a^\dagger a + \frac{1}{2} \sigma_3$ is conserved. In the Rabi model, although Q is not conserved, it only changes by a bounded

amount: $|\Delta Q| \leq 1$. We can see this by defining $|q\rangle$ as a basis in which Q is block diagonal, so that $\langle q'|H|q\rangle = 0$ if $\Delta Q = |q - q'| > 1$. This makes the Hamiltonian block-tridiagonal and we solve it in terms of continued fractions. Unlike Braak, who was interested in the integrability of the system, we ask another fundamental question "What makes a quantum mechanical system solvable?"

To view this problem from a different perspective we look at the Hamiltonian

$$H_L = \omega L_3 + \Delta R_3 + g L_1 R_1, \quad (2)$$

where L and R are angular momentum matrices with magnitude l and r respectively and we have absorbed a factor of 2 into g . This looks eerily similar to the Rabi model and in section V we use a contraction of the algebra to show that in the limit as $l \rightarrow \infty$ with $r = \frac{1}{2}$, (2) becomes the Rabi Hamiltonian (1). The advantage of studying this new, more general Hamiltonian, is that unlike the Rabi model it has finite dimension.

To find the matrix elements of H_L we use the basis states

$$|\psi_L\rangle = |l, m_l, r, m_r\rangle = |l, m_l\rangle \otimes |r, m_r\rangle \quad (3)$$

where $m_l = -l, -l+1, \dots, l$ and $m_r = -r, -r+1, \dots, r$ are the azimuthal components of angular momenta in the L_3, R_3 basis. This allows us to write the Hamiltonian as a block tridiagonal matrix

$$H_L = \begin{pmatrix} A_{-l} & B_{-l+1} & 0 & \cdots & 0 \\ B_{-l+1} & A_{-l+1} & B_{-l+2} & \cdots & 0 \\ 0 & B_{-l+2} & A_{-l+2} & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & B_l \\ 0 & 0 & \cdots & B_l & A_l \end{pmatrix} \quad (4)$$

with

$$\begin{aligned} A_k &= k\omega I_r + \Delta R_3 \\ B_k &= \sqrt{l(l+1) - k(k-1)} g R_1, \end{aligned} \quad (5)$$

where I_r is an r dimensional identity matrix.

II. CONTINUED FRACTIONS AND TRIDIAGONAL MATRICES

The general theory of analytical continued fractions was developed by Stieltjes in the late 19th century while

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studying divergent power series[6]. These continued fractions are of the form

$$f(z) = \frac{1}{z + a_1 - \frac{b_1^2}{z + a_2 - \frac{b_2^2}{z + a_3 - \dots}}} \quad (6)$$

and are intimately connected with ordinary tridiagonal matrices of the form

$$A = \begin{pmatrix} a_0 & b_1 & 0 & 0 & \cdot \\ c_1 & a_1 & b_2 & 0 & \cdot \\ 0 & c_2 & a_2 & b_3 & \cdot \\ 0 & 0 & c_3 & a_3 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}. \quad (7)$$

This connection can be seen by defining

$$\begin{aligned} \Delta_{-1}(z) &\equiv 1 \\ \Delta_0(z) &= a_0 - z \\ \Delta_k(z) &= (a_k - z) \Delta_{k-1}(z) - b_k c_k \Delta_{k-2}(z) \end{aligned} \quad (8)$$

which gives us a continued fraction

$$\begin{aligned} S_k(z) &= \frac{\Delta_k(z)}{\Delta_{k-1}(z)} \\ &= (a_k - z) - \frac{b_k c_k}{S_{k-1}(z)} \end{aligned} \quad (9)$$

whose roots are the eigenvalues of the matrix. If however, A is an infinite dimensional matrix, the roots of S_k represent the k^{th} approximation to the eigenvalues of A . When a_k are real and $b_k c_k > 0$ the function S_k is a Sturm sequence, meaning the zeros are real and the roots of S_{k-1} are the poles of S_k . This gives us an easy way to calculate higher approximations to the eigenvalues: between any two pairs of poles of S_k is an eigenvalue.

This tells us that as long as S_k is a convergent continued fraction, even if the matrix it represents is infinite dimensional, it can still be solved to the desired level of precision. More importantly since the convergence of such continued fractions has been well established for over a hundred years this method gives a true check as to whether or not diagonalizing increasingly larger matrices will converge to the eigenvalues of an infinite dimensional matrix[6].

III. EIGENVALUES OF BLOCK TRIDIAGONAL MATRICES

The usefulness of tridiagonal form to prove convergence and the block tridiagonal form of H_L prompts us to ask “is it also possible to develop relations similar to (8) and (9) for block tridiagonal matrices?” The answer is, in some cases, yes. Using the transfer matrix method of Molinari [7] we can find the eigenvalues of an

$(n+1) \times (n+1)$ block tridiagonal matrix

$$M = \begin{pmatrix} A_0 & B_1 & & & \\ B_1 & A_1 & B_2 & & \\ & B_2 & A_2 & \ddots & \\ & & \ddots & \ddots & B_n \\ & & & B_n & A_n \end{pmatrix}, \quad (10)$$

where A_k, B_k are $m \times m$ square matrices, and A_k is of the form $A'_k - zI_m$, where z are the eigenvalues. In this case $\det M = 0$, allowing us to set

$$M\Psi = \begin{pmatrix} A_0 & B_1 & 0 & 0 & \cdot \\ C_1 & A_1 & B_2 & 0 & \cdot \\ 0 & C_2 & A_2 & B_3 & \cdot \\ 0 & 0 & C_3 & A_3 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \\ \cdot \end{pmatrix} = 0 \quad (11)$$

where Ψ is a null vector with components $\psi_k \in \mathbb{C}^m$, giving us a set of equations

$$\begin{aligned} A_0 \psi_0 + B_1 \psi_1 &= 0 \\ B_{k+1} \psi_{k+1} + A_k \psi_k + C_k \psi_{k-1} &= 0 \\ A_n \psi_n + C_n \psi_{n+1} &= 0. \end{aligned} \quad (12)$$

This can be written recursively as

$$\begin{bmatrix} \psi_{k+1} \\ \psi_k \end{bmatrix} = \begin{bmatrix} -B_{k+1}^{-1} A_k & -B_{k+1}^{-1} C_k \\ I_m & 0 \end{bmatrix} \begin{bmatrix} \psi_k \\ \psi_{k-1} \end{bmatrix}, \quad (13)$$

which defines the transfer matrix

$$\begin{aligned} T_k &= \begin{bmatrix} -B_{k+1}^{-1} A_k & -B_{k+1}^{-1} C_k \\ I_m & 0 \end{bmatrix} T_{k-1} \\ T_n &= \begin{bmatrix} A_n & C_n \\ I_m & 0 \end{bmatrix} \begin{bmatrix} -B_n^{-1} A_{n-1} & -B_n^{-1} C_{n-1} \\ I_m & 0 \end{bmatrix} \dots \\ &\times \begin{bmatrix} -B_1^{-1} A_0 & -B_1^{-1} C_0 \\ I_m & 0 \end{bmatrix}. \end{aligned} \quad (14)$$

If we define the top left element of T_k as

$$T_{k,11} = -B_{k+1}^{-1} A_k T_{k-1,11} - B_{k+1}^{-1} C_k T_{k-2,11}, \quad (15)$$

since $\psi_{n+2} = 0$ and $\psi_{-1} = 0$

$$\begin{bmatrix} 0 \\ \psi_{n+1} \end{bmatrix} = T_n \begin{bmatrix} \psi_1 \\ 0 \end{bmatrix} \quad (16)$$

so $\det T_{n,11} = 0$ is the same as $\det M = 0$, defining the eigenvalue equation.

IV. SPECTRUM OF H_L

For the most general form of H_L , integer values of r correspond to singular matrices that have no inverse, so the above method can be modified to the form of Salkuyeh[8]. It is possible to solve H_L for any half integer spin but the most elegant case, which is also interesting

because of it's ties to the Rabi model H_R , is for $r = \frac{1}{2}$. In this case

$$\begin{aligned} R_i &= \frac{1}{2}\sigma_i \\ A_k &= kI + \frac{\Delta}{2}\sigma_3 \\ B_k &= C_k = g\sqrt{l(l+1) - k(k-1)}\sigma_1 \\ &= b_k\sigma_1 \\ B_k^{-1} &= b_k^{-1}\sigma_1 \end{aligned} \quad (17)$$

where we have absorbed a factor of $\frac{1}{2}$ into g and defined $b_k = g\sqrt{l(l+1) - k(k-1)}$. This gives us the eigenvalue equation $\det(H_L - zI) = 0 = \det(T_{11}(z))$, where the transfer matrix can be simplified by multiplying each 2×2 matrix in T_k by b_k to give

$$\begin{aligned} T_k &= \begin{bmatrix} \sigma_1(A_k - zI) & b_k I \\ -b_k & 0 \end{bmatrix} T_{k-1} \\ T_{k,11} &= \sigma_1 A_k T_{k-1,11} - b_m^2 T_{k-2,11} \\ &= \left((k\omega - z)\sigma_1 - i\frac{\Delta}{2}\sigma_2 \right) T_{k-1,11} - B_k^2 T_{k-2,11}. \end{aligned} \quad (18)$$

Due to the special property of Pauli matrices $\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$ we see that if we define

$$S_m = T_{m,11} T_{m-1,11}^{-1} \quad (19)$$

we have the recursive matrix

$$S_m = (m\omega - z)\sigma_1 - i\Delta\sigma_2 - B_m^2 S_{m-1}^{-1}, \quad (20)$$

which is of the form

$$S_m = \begin{pmatrix} 0 & a(z) \\ b(z) & 0 \end{pmatrix}, \quad (21)$$

with inverse

$$S_m^{-1} = \begin{pmatrix} 0 & \frac{1}{b(z)} \\ \frac{1}{a(z)} & 0 \end{pmatrix}. \quad (22)$$

This allows us to write S_m in matrix form

$$S_m = \begin{pmatrix} 0 & a_m^+ - \frac{b_m^2}{a_{m-1}^- - \frac{b_{m-1}^2}{a_{m-2}^+ - \dots}} \\ a_m^- - \frac{b_m^2}{a_{m-1}^+ - \frac{b_{m-1}^2}{a_{m-2}^- - \dots}} & 0 \end{pmatrix}, \quad (23)$$

where $a_m^\pm = m\omega - z \pm \frac{\Delta}{2}$, and by setting $S_m = 0$ we obtain two terminating continued fractions

$$S_{m,\pm}(z) = m\omega - z \mp (-1)^m \frac{\Delta}{2} - \frac{b_m^2}{S_{m-1,\pm}(z)}, \quad (24)$$

where $m = -l, -l+1, \dots, l$, whose zeros are the eigenvalues of the H_L . Because these are finite dimensional matrices, calculating the roots of $S_{l,+}$ gives the even parity spectrum while the roots of $S_{l,-}$ give the odd parity spectrum.

V. CONTRACTION OF H_L TO H_R

We could of course perform the same procedure on the Rabi model H_R to find it's eigenvalues in a similar way but a deeper connection between the two systems can be seen by performing a singular change of basis. It was discovered by Inonu and Wigner [9] that a transformation of this type changes one Lie Algebra into another using a process called contraction. We briefly summarize Gilmore's [10] description of a different class of contractions to show the relationship between (1) and (2).

A Lie algebra defined by the basis vectors X_i is closed under commutation, so the commutators

$$[X_i, X_j] = C_{ij}^k X_k \quad (25)$$

are contained in the algebra. The structure constants C_{ij}^k completely determine the algebra, however it is possible to perform a change of basis transformation

$$Y_i = M_i^j X_j \quad (26)$$

where the new structure constant $C'_{ij}{}^k$ becomes

$$C'_{ij}{}^k = (M^{-1})_i^l (M^{-1})_j^m C_{lm}^n M_n^k \quad (27)$$

due to a non-singular transformation. If we allow the transformation to be parameter dependent, where

$$\begin{aligned} Y_i &= M_i^j(\epsilon) X_j \\ C'_{ij}{}^k &= C_{ij}^k(\epsilon) \end{aligned} \quad (28)$$

the structure constant often converges to a new Lie Algebra if $C_{ij}^k(\epsilon)$ becomes singular in the limit as $\epsilon \rightarrow \infty$.

One representation of angular momentum is the compact unitary group $U(2)$, which is spanned by the operators J_3, J_\pm, J_0 , (J_0 is the identity) which correspond to the commutation relations

$$\begin{aligned} [J_3, J_\pm] &= \pm J_\pm \\ [J_+, J_-] &= 2J_3 \\ [J_0, \mathbf{J}] &= 0. \end{aligned} \quad (29)$$

We can now change basis to the Heisenberg group H_4 by using

$$\begin{pmatrix} h_+ \\ h_- \\ h_3 \\ h_0 \end{pmatrix} = \begin{pmatrix} c & & & \\ & c & & \\ & & 1 & \frac{1}{2c^2} \\ & & & 1 \end{pmatrix} \begin{pmatrix} J_+ \\ J_- \\ J_3 \\ J_0 \end{pmatrix} \quad (30)$$

which gives us

$$\begin{aligned} [h_3, h_\pm] &= \pm h_\pm \\ [h_+, h_-] &= 2c^2 h_3 - h_0 \\ [h_0, \mathbf{h}] &= 0 \end{aligned} \quad (31)$$

and in the limit as $c \rightarrow 0$

$$\begin{aligned} [h_3, h_\pm] &= \pm h_\pm \\ [h_+, h_-] &= -h_0 = -I \end{aligned} \quad (32)$$

which satisfy the same commutations as the single mode photon operators

$$\begin{aligned} [N = a^\dagger a, a] &= -a \\ [N = a^\dagger a, a^\dagger] &= a^\dagger \\ [a^\dagger, a] &= -1. \end{aligned} \quad (33)$$

We can now identify (in the limit as $c \rightarrow 0$)

$$\begin{aligned} h_3 &= N \\ h_+ &= a^\dagger \\ h_- &= a. \end{aligned} \quad (34)$$

To see how the basis states change we first operate h_3 on the angular momentum state $|j, m\rangle$ to get

$$h_3 |j, m\rangle = \left(J_3 + \frac{1}{2c^2} J_0\right) |j, m\rangle = \left(m + \frac{1}{2c^2}\right) |j, m\rangle. \quad (35)$$

The ground state of this system corresponds to $m = -j$, so the n^{th} state is $n = j + m$. In order for the limit to be well defined we require

$$\lim_{c \rightarrow 0} \left(m + \frac{1}{2c^2}\right) = \lim_{c \rightarrow 0} \left(n - j + \frac{1}{2c^2}\right) \quad (36)$$

to also be well defined. We have already equated h_3 with the number operator, so the requirement becomes

$$\lim_{c \rightarrow 0} \left(-j + \frac{1}{2c^2}\right) = 0 \quad (37)$$

telling us that $2jc^2 = 1$, or in other words as $c \rightarrow 0$, $j \rightarrow \infty$, and

$$\lim_{\substack{c \rightarrow 0 \\ j \rightarrow \infty}} h_3 |j, m\rangle = n | \infty, n \rangle. \quad (38)$$

Similarly, we see that

$$\begin{aligned} a^\dagger |n\rangle &= \lim_{c \rightarrow 0} c J_+ |j, m\rangle \\ &= \lim_{c \rightarrow 0} \sqrt{j(j+1) - m(m+1)} |j, m+1\rangle \\ &= \lim_{c \rightarrow 0} \sqrt{(1 - c^2 n)(n+1)} |j, m+1\rangle \\ &= \sqrt{n+1} |n+1\rangle. \end{aligned} \quad (39)$$

So a contraction on H_L changes (24) to

$$S_{k,\pm}(z) = k\omega - z \mp (-1)^k \frac{\Delta}{2} - \frac{g^2 k}{S_{k-1,\pm}(z)}, \quad (40)$$

$k = 0, 1, 2, 3, \dots$. Its zeros give us the spectrum of the Rabi model H_R . As discussed in section II, there is a rapidly convergent algorithm to find the zeros, which takes advantage of the fact that there is a zero of $S_{k,\pm}(z)$ in between two of its poles, which are simply the zeroes of the previous approximation, $S_{k-1,\pm}(z)$. Thus we can limit the search for each zero to these intervals, increasing the size k of the matrix in each step.

VI. CONCLUSION

We show that the Rabi model can be thought of as the limit of a sequence of finite dimensional block tridiagonal Hamiltonians, each of which can be solved by a continued fraction method. The solvability of the Rabi model can thus be understood as due to this finite dimensional truncation and the existence of approximate conservation laws (selection rules for transition matrix elements) that ensure block tridiagonality, explaining why even a Hamiltonian with a broken symmetry can be solved [3]. Conversely, we should expect that systems which do not allow convergent finite dimensional approximations exhibit quantum chaos. We hope to construct such an example in a later publication.

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